Double and Half Angle Identities

-additional tools to use in order to find exact values of trigonometric functions, or to simply expressions involving trigonometric functions.

We can use sum and difference identities to derive all 3 Double Angle Identities.

Expand and simplify each of the following:

$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$\boxed{\sin(2\alpha) = 2\sin\alpha\cos\alpha}$

$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\boxed{\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha}$

$1-\sin^2\alpha \qquad 1-\cos^2\alpha$

$\boxed{\cos(2\alpha) = 1 - 2\sin^2\alpha}$ $\qquad$ $\boxed{\cos(2\alpha) = 2\cos^2\alpha - 1}$

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

$$\boxed{\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

Half Angle Identities

$$\boxed{\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}}$$

$$\boxed{\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}}$$

**Signs of sine and cosine depend on the quadrant in which $\frac{u}{2}$ lies.*

$$\boxed{\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}}$$

or

$$\boxed{\tan\frac{u}{2} = \frac{\sin u}{1+\cos u}}$$

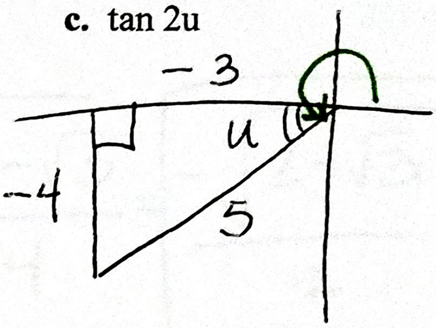
Ex 1:

Given $\cos u = -\frac{3}{5}$ with $\pi < u < \frac{3\pi}{2}$, find the following.

$\tan u = \frac{4}{3}$

$\sin u = \frac{-4}{5}$

a. $\sin 2u$ **b.** $\sin \frac{u}{2}$ **c.** $\tan 2u$



a) $\sin(2u) = 2\sin u \cos u$

$$= \frac{2}{1}\left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right)$$

$$= \boxed{\frac{24}{25}}$$

b) $\sin \frac{u}{2} = +\sqrt{\frac{1-\cos u}{2}}$

$$= \sqrt{\frac{1+\left(+\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(1+\frac{3}{5}\right)5}{(2)5}} = \sqrt{\frac{5+3}{10}}$$

$$= \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \boxed{\frac{2\sqrt{5}}{5}}$$

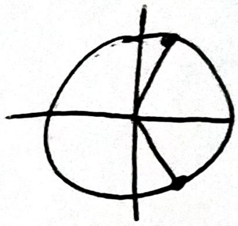
c) $\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$

$$= \frac{\frac{2}{1}\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\left(\frac{8}{\cancel{3}}\right)\frac{\cancel{9}^{3}}{1}}{\left(1-\frac{16}{\cancel{9}}\right)9} = \frac{24}{9-16} = \frac{24}{-7} = \boxed{-\frac{24}{7}}$$

Ex 2: Use a half angle identity to find the exact value of cos 105°. $= \frac{210^\circ}{2} \leftarrow u$

$$\cos\frac{210^\circ}{2} = -\sqrt{\frac{1+\cos 210^\circ}{2}}$$

$$= -\sqrt{\frac{\left(1+\left(-\frac{\sqrt{3}}{2}\right)\right)(2)}{(2)(2)}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{-\frac{\sqrt{2-\sqrt{3}}}{2}}$$



Ex 3: Use double angle identities to solve each of the following equations

a) $\cos 2x + \cos x = 0$ for all values of x.

$u = \cos x$

$$2\cos^2 x - 1 + \cos x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$2u^2 + u - 1 = 0$$

$$(2u-1)(u+1) = 0$$

$$2u - 1 = 0 \qquad u + 1 = 0$$

$$u = \frac{1}{2} \qquad u = -1$$

$$\cos x = \frac{1}{2}$$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3} + 2\pi n}$$

$$\cos x = -1$$

$$\boxed{x = \pi + 2\pi n}$$

b) $\tan 2x + \tan x = 0$ for $0 \le x < 2\pi$.

$$(1-\tan^2 x)\left(\frac{2\tan x}{1-\tan^2 x}\right) + \tan x = 0\,(1-\tan^2 x)$$

$$2\tan x + \tan x(1-\tan^2 x) = 0$$

$$2\tan x + \tan x - \tan^3 x = 0$$

$u = \tan x$

$$3\tan x - \tan^3 x = 0$$

$$3u - u^3 = 0$$

$$u(3-u^2) = 0$$

$$u = 0 \qquad 3 - u^2 = 0$$

$$u^2 = 3 \rightarrow u = \pm\sqrt{3}$$

$$\tan x = 0$$

$$\boxed{x = 0}$$

$$\tan x = \pm\sqrt{3}$$

$$\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$